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On the spatial behavior in the theory of swelling porous elastic soils

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Abstract

This paper is concerned with the study of the spatial behavior of the processes associated with a mixture consisting of three components: an elastic solid, a viscous fluid and a gas. An appropriate time-weighted surface power function is used in order to describe the spatial behavior of the processes in question. Spatial estimates of Saint–Venant type (for bounded bodies) and Phragmèn–Lindelöf type (for unbounded bodies) with time-dependent and time-independent rates are established. For unbounded bodies the asymptotic spatial behavior of the processes is also studied by means of an appropriate volumetric measure.

Keywords: Spatial behavior; Time-dependent; Asymptotic

1. Introduction

The continuum theory of mixtures are extensively studied in literature. A presentation of the work on the subject can be found in review articles by Bowen (1976), Atkin and Craine (1976a,b), and Bedford and Drumheller (1983).

Eringen (1994) pointed out the importance of the theory of mixtures to the applied field of swelling. In this connection Eringen (1994) has developed a continuum theory of swelling porous elastic soils as a continuum theory of mixture for porous elastic solids filled with fluid and gas. The theory provides a fundamental basis for the treatment of various practical problems in the field of swelling, oil exploration, slurries and consolidation problems. The theory is relevant to problems in the oil exploration industry, since oil is viscous and is usually accompanied by gas in underground rocks, porous solid in slurries and muddy river beds.

In the context of theory of swelling porous elastic soils some continuous dependence and uniqueness results have been established by Galeş (2002).

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This article is concerned with the study of spatial behavior in the isothermal linear theory of swelling porous elastic soils.

The study of spatial decay and growth of solutions of time-dependent problems is of relatively recent origin (see e.g. Edelstein, 1969; Knowles, 1971; Flavin and Knops, 1987; Ieşan and Quintanilla, 1995; Chiriţă and Ciarletta, 1999; Chiriţă and Scalia, 2001 and the references cited by Horgan and Knowles, 1983; Horgan, 1989). Chiriţă and Ciarletta (1999) were presented a method for the study of spatial behavior in dynamics of continua. The method is based on a set of properties for an appropriate time-weighted surface power function associated with the dynamic processes. In linear elastodynamics and viscoelastodynamics, there are obtained spatial decay estimates with time-independent decay rate inside of the domain of influence, while for models which implies the presence of a dissipation energy (see also Chiriţă and Danescu, 2000; Chiriţă and Scalia, 2001), there are obtained spatial estimates characterized by independent as well as time-dependent decay and growth rates.

Our analysis in the present paper is based on the results obtained by Chiriţă and Ciarletta (1999) and Chiriţă and Scalia (2001). Since for the model in question there exist a dissipation energy we obtain spatial decay and growth estimates characterized by independent and time-dependent rates. Thus, for bounded bodies we establish spatial decay estimates of Saint–Venant type, while for unbounded bodies we establish some alternatives of Phragmèn–Lindelöf type. We also outline a class of mixtures for which we can improve the spatial decay estimates by studying the asymptotic behavior of the processes by means of an appropriate volumetric measure. A similar measure has been used by Scalia (2002) to study the asymptotic spatial behavior in linear thermoelasticity of materials with voids. The results are obtained under positive definiteness assumption upon the internal energy density.

The plan of the paper is as follows: In Section 2 we set down the basic equations and we discuss some restrictions upon the constitutive coefficients. Section 3 contains the derivation of some general properties of an appropriate time-weighted surface power function associated with the mixture and some results that describe the spatial behavior of processes for bounded and unbounded bodies. Section 4 examines the asymptotic spatial behavior of processes.

2. Basic equation—some preliminary results

We refer the motion of a continuum to a fixed system of rectangular Cartesian axes $0x_k$ ($k = 1, 2, 3$). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over integer (1, 2, 3), summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation.

We consider a body that at time $t = 0$ occupies the bounded or unbounded regular region B of Euclidean three-dimensional space whose boundary is the regular surface ∂B.

We assume that B is occupied by a mixture consisting of three components: an elastic solid, a viscous fluid and a gas. We use superscripts s, f, g to denote respectively, the elastic solid, the fluid and the gas. Let ρ_0^{s}, ρ_0^{f} and ρ_0^{g} denote the densities at time $t = 0$ of the three constituents, respectively. We consider the fundamental equations for mechanical behavior of the mixture in the framework of the linearized theory (see Eringen, 1994; Galeş, 2002). Thus, the equations of motion are

$$\begin{aligned} &t_{ji,j}^{\mathrm{s}} + \rho_0^{\mathrm{s}} f_i^{\mathrm{s}} + p_i^{\mathrm{f}} + p_i^{\mathrm{g}} = \rho_0^{\mathrm{s}} \ddot{u}_i^{\mathrm{s}}, \\ &t_{ji,j}^{\mathrm{f}} + \rho_0^{\mathrm{f}} f_i^{\mathrm{f}} - p_i^{\mathrm{f}} = \rho_0^{\mathrm{f}} \ddot{u}_i^{\mathrm{f}}, \\ &t_{ji,j}^{\mathrm{g}} + \rho_0^{\mathrm{g}} f_i^{\mathrm{g}} - p_i^{\mathrm{g}} = \rho_0^{\mathrm{g}} \ddot{u}_i^{\mathrm{g}}, \quad \text{in } B \times [0, \infty), \end{aligned} \tag{1}$$

where t_{ij}^{s}, t_{ij}^{f} and t_{ij}^{g} are the partial stress tensors, f_i^{s}, f_i^{f} and f_i^{g} are the body forces, u_i^{s}, u_i^{f} and u_i^{g} are the displacement vector fields, and p_i^{f} and p_i^{g} are the internal body forces.

The constitutive equations for a homogeneous and isotropic mixture are

$$\begin{aligned} t^{\rm s}_{ij} &= \left(-\sum_{a={\rm f,g}} \sigma^a e^a_{rr} + \lambda e^{\rm s}_{rr} \right)\delta_{ij} + 2\mu e^{\rm s}_{ij}, \\ t^{\rm f}_{ij} &= \left(-\sigma^{\rm f} e^{\rm s}_{rr} - \sum_{a={\rm f,g}} \sigma^{fa} e^a_{rr} + \lambda_v \dot{e}^{\rm f}_{rr} \right)\delta_{ij} + 2\mu_v \dot{e}^{\rm f}_{ij}, \\ t^{\rm g}_{ij} &= \left(-\sigma^{\rm g} e^{\rm s}_{rr} - \sum_{a={\rm f,g}} \sigma^{ga} e^a_{rr} \right)\delta_{ij}, \\ p^a_i &= \sum_{b={\rm f,g}} \xi^{ab}(\dot{u}^b_i - \dot{u}^{\rm s}_i), \quad a = {\rm f,g} \ \text{in } \overline{B}\times[0,\infty), \end{aligned} \tag{2}$$

where σ^a $(a = {\rm f,g})$, λ, μ, σ^{ab} $(a,b = {\rm f,g})$, λ_v, μ_v, ξ^{ab} $(a,b = {\rm f,g})$ are constitutive constants; δ_{ij} is the Kronecker delta; and $e^{\rm s}_{ij}$, $e^{\rm f}_{ij}$ and $e^{\rm g}_{ij}$ are defined by

$$\begin{aligned} e^{\rm s}_{ij} &= \tfrac{1}{2}(u^{\rm s}_{i,j} + u^{\rm s}_{j,i}), \\ e^{\rm f}_{ij} &= \tfrac{1}{2}(u^{\rm f}_{i,j} + u^{\rm f}_{j,i}), \\ e^{\rm g}_{ij} &= \tfrac{1}{2}(u^{\rm g}_{i,j} + u^{\rm g}_{j,i}), \quad \text{in } \overline{B}\times[0,\infty). \end{aligned} \tag{3}$$

The coefficients in (2) have the following symmetries:

$$\sigma^{ab} = \sigma^{ba}, \qquad \xi^{ab} = \xi^{ba}, \qquad a,b = {\rm f,g}. \tag{4}$$

Let M and N be non-negative integers. We say that h is of class $C^{M,N}$ on $\overline{B}\times[0,\infty)$ if h is continuous on $\overline{B}\times[0,\infty)$, and the functions

$$\frac{\partial^m}{\partial x_i \partial x_j \cdots \partial x_r}\left(\frac{\partial^n h}{\partial t^n}\right), \quad m \in \{0,1,\ldots,M\}, \ n \in \{0,1,\ldots,N\}, \ m+n \leqslant \max\{M,N\},$$

exist and are continuous on $\overline{B}\times[0,\infty)$. We denote $C^{M,M}$ by C^M.

Throughout this paper by an admissible process we mean the ordered array $\mathscr{P} = \{\mathbf{u}^{\rm s},\mathbf{u}^{\rm f},\mathbf{u}^{\rm g};\ \mathbf{e}^{\rm s},\mathbf{e}^{\rm f},\mathbf{e}^{\rm g};\ \mathbf{t}^{\rm s},\mathbf{t}^{\rm f},\mathbf{t}^{\rm g};\ \mathbf{p}^{\rm f},\mathbf{p}^{\rm g}\}$ with the properties

(a) $u^{\rm s}_i$, $u^{\rm f}_i$, $u^{\rm g}_i$ are of class $C^{1,2}$ on $\overline{B}\times[0,\infty)$;
(b) the symmetric fields $e^{\rm s}_{ij}$, $e^{\rm f}_{ij}$, $e^{\rm g}_{ij}$ are of class $C^{0,1}$ on $\overline{B}\times[0,\infty)$;
(c) the symmetric fields $t^{\rm s}_{ij}$, $t^{\rm f}_{ij}$, $t^{\rm g}_{ij}$ are of class $C^{1,0}$ on $\overline{B}\times[0,\infty)$;
(d) $p^{\rm f}_i$, $p^{\rm g}_i$ are of class C^0 on $\overline{B}\times[0,\infty)$.

Further, we say that $\mathscr{P} = \{\mathbf{u}^{\rm s},\mathbf{u}^{\rm f},\mathbf{u}^{\rm g};\ \mathbf{e}^{\rm s},\mathbf{e}^{\rm f},\mathbf{e}^{\rm g};\ \mathbf{t}^{\rm s},\mathbf{t}^{\rm f},\mathbf{t}^{\rm g};\ \mathbf{p}^{\rm f},\mathbf{p}^{\rm g}\}$ is a dynamic process for the mixture corresponding to body forces $\mathbf{f}^{\rm s}$, $\mathbf{f}^{\rm f}$ and $\mathbf{f}^{\rm g}$ if $\mathscr{P}$ is an admissible process and satisfies the basic equations (1)–(3). To the dynamic process $\mathscr{P}$ we associate the surface tractions s^α_i $(\alpha = {\rm s,f,g})$ defined at every regular point of a boundary surface by

$$s^\alpha_i(\mathbf{x},t) = t^\alpha_{ji}(\mathbf{x},t)n_j(\mathbf{x}), \quad \alpha = {\rm s,f,g}, \tag{5}$$

where n_j are the components of the outward unit normal vector to the boundary surface of a region. We call the array $\mathscr{F} = \{\mathbf{f}^{\rm s},\mathbf{f}^{\rm f},\mathbf{f}^{\rm g};\mathbf{s}^{\rm s},\mathbf{s}^{\rm f},\mathbf{s}^{\rm g}\}$ the external force system for $\mathscr{P}$.

As it was shown by Eringen (1994), the local form of the Clausius–Duhem inequality implies

$$3\lambda_v + 2\mu_v \geqslant 0, \quad \mu_v \geqslant 0, \tag{6}$$

and the positive semi-definiteness of the following symmetric matrix:

$$\mathbf{\Delta} = \begin{pmatrix} \xi^{\mathrm{ff}} & \xi^{\mathrm{fg}} \\ \xi^{\mathrm{fg}} & \xi^{\mathrm{gg}} \end{pmatrix}, \tag{7}$$

so that the dissipation energy density $\mathbf{\Phi}$, corresponding to the displacement vectors $\mathbf{u} = [\mathbf{u}^{\mathrm{s}}, \mathbf{u}^{\mathrm{f}}, \mathbf{u}^{\mathrm{g}}]$ and defined by

$$\mathbf{\Phi}(\mathbf{u}) = \lambda_v \dot{e}^{\mathrm{f}}_{ii}(\mathbf{u})\dot{e}^{\mathrm{f}}_{jj}(\mathbf{u}) + 2\mu_v \dot{e}^{\mathrm{f}}_{ij}(\mathbf{u})\dot{e}^{\mathrm{f}}_{ij}(\mathbf{u}) + \sum_{a,b=\mathrm{f,g}} \xi^{ab}(\dot{u}^a_i - \dot{u}^{\mathrm{s}}_i)(\dot{u}^b_i - \dot{u}^{\mathrm{s}}_i), \tag{8}$$

is non-negative.

The internal energy density $\mathscr{E}$ corresponding to the displacement vectors $\mathbf{u} = [\mathbf{u}^{\mathrm{s}}, \mathbf{u}^{\mathrm{f}}, \mathbf{u}^{\mathrm{g}}]$ is defined by

$$\mathscr{E} = \frac{1}{2}\lambda e^{\mathrm{s}}_{ii}(\mathbf{u})e^{\mathrm{s}}_{jj}(\mathbf{u}) + \mu e^{\mathrm{s}}_{ij}(\mathbf{u})e^{\mathrm{s}}_{ij}(\mathbf{u}) - \sum_{a=\mathrm{f,g}} \sigma^a e^a_{ii}(\mathbf{u})e^{\mathrm{s}}_{jj}(\mathbf{u}) - \frac{1}{2}\sum_{a,b=\mathrm{f,g}} \sigma^{ab} e^a_{ii}(\mathbf{u})e^b_{jj}(\mathbf{u}). \tag{9}$$

Throughout this paper we shall assume that the following symmetric matrix is positive definite:

$$\delta = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & -\sigma^{\mathrm{f}} & -\sigma^{\mathrm{g}} \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & -\sigma^{\mathrm{f}} & -\sigma^{\mathrm{g}} \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & -\sigma^{\mathrm{f}} & -\sigma^{\mathrm{g}} \\ 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 \\ -\sigma^{\mathrm{f}} & -\sigma^{\mathrm{f}} & -\sigma^{\mathrm{f}} & 0 & 0 & 0 & -\sigma^{\mathrm{ff}} & -\sigma^{\mathrm{fg}} \\ -\sigma^{\mathrm{g}} & -\sigma^{\mathrm{g}} & -\sigma^{\mathrm{g}} & 0 & 0 & 0 & -\sigma^{\mathrm{fg}} & -\sigma^{\mathrm{gg}} \end{pmatrix}. \tag{10}$$

Thus, the internal energy density $\mathscr{E}(\mathbf{u})$ is a positive definite quadratic form in terms of $e^{\mathrm{s}}_{ij}(\mathbf{u})$, $e^{\mathrm{f}}_{ii}(\mathbf{u})$ and $e^{\mathrm{g}}_{ii}(\mathbf{u})$. Then, we have

$$\lambda e^{\mathrm{s}}_{ii}e^{\mathrm{s}}_{jj} + 2\mu e^{\mathrm{s}}_{ij}e^{\mathrm{s}}_{ij} - 2\sum_{a=\mathrm{f,g}} \sigma^a e^a_{ii}e^{\mathrm{s}}_{jj} - \sum_{a,b=\mathrm{f,g}} \sigma^{ab} e^a_{ii}e^b_{jj} \leqslant \sigma_{\mathrm{M}}(e^{\mathrm{s}}_{ij}e^{\mathrm{s}}_{ij} + e^{\mathrm{f}}_{ij}e^{\mathrm{f}}_{ij} + e^{\mathrm{g}}_{ij}e^{\mathrm{g}}_{ij}), \tag{11}$$

where

$$\sigma_{\mathrm{M}} = 3\delta_{\mathrm{M}}, \tag{12}$$

and δ_{M} is the largest eigenvalue of the matrix δ. In order to get the relation (11) we have been used the relation

$$e^a_{ii}(\mathbf{u})e^a_{jj}(\mathbf{u}) \leqslant 3e^a_{ij}(\mathbf{u})e^a_{ij}(\mathbf{u}), \quad a = \mathrm{f,g}. \tag{13}$$

Lemma 1. *Let $\mathscr{P}$ be an admissible process for the mixture satisfying the constitutive equations* (2). *Then, for every positive ϵ_1 we have*

$$\sum_{\alpha=\mathrm{s,f,g}} t^\alpha_{ij}(\mathbf{u})t^\alpha_{ij}(\mathbf{u}) \leqslant 2(1+\epsilon_1)\sigma_{\mathrm{M}}\mathscr{E}(\mathbf{u}) + \left(1 + \frac{1}{\epsilon_1}\right)\mu^v_{\mathrm{M}}\left[\lambda_v \dot{e}^{\mathrm{f}}_{ii}(\mathbf{u})\dot{e}^{\mathrm{f}}_{jj}(\mathbf{u}) + 2\mu_v \dot{e}^{\mathrm{f}}_{ij}(\mathbf{u})\dot{e}^{\mathrm{f}}_{ij}(\mathbf{u})\right], \tag{14}$$

where

$$\mu^v_{\mathrm{M}} = \max\{2\mu_v, 3\lambda_v + 2\mu_v\}. \tag{15}$$

Proof. From the relations (2)–(4), (6), (10) and (11) we deduce

$$\sum_{\alpha=\mathrm{s,f,g}} t_{ij}^{\alpha}(\mathbf{u})t_{ij}^{\alpha}(\mathbf{u}) = \lambda e_{ii}^{\mathrm{s}}(\mathbf{u})t_{jj}^{\mathrm{s}}(\mathbf{u}) + 2\mu e_{ij}^{\mathrm{s}}(\mathbf{u})t_{ij}^{\mathrm{s}}(\mathbf{u}) - \sum_{a=\mathrm{f,g}} \sigma^{a} e_{ii}^{a}(\mathbf{u})t_{jj}^{\mathrm{s}}(\mathbf{u}) - \sum_{a=\mathrm{f,g}} \sigma^{a} e_{ii}^{\mathrm{s}}(\mathbf{u})t_{jj}^{a}(\mathbf{u}) - \sum_{a=\mathrm{f,g}} \sigma^{ab} e_{ii}^{a}(\mathbf{u})t_{jj}^{b}(\mathbf{u})$$
$$+ \lambda_{v}\dot{e}_{ii}^{\mathrm{f}}(\mathbf{u})t_{jj}^{\mathrm{f}}(\mathbf{u}) + 2\mu_{v}\dot{e}_{ij}^{\mathrm{f}}(\mathbf{u})t_{ij}^{\mathrm{f}}(\mathbf{u}) \leqslant \left[\lambda e_{ii}^{\mathrm{s}}(\mathbf{u})e_{jj}^{\mathrm{s}}(\mathbf{u}) + 2\mu e_{ij}^{\mathrm{s}}(\mathbf{u})e_{ij}^{\mathrm{s}}(\mathbf{u}) - 2\sum_{a=\mathrm{f,g}} \sigma^{a} e_{ii}^{a}(\mathbf{u})e_{jj}^{\mathrm{s}}(\mathbf{u})\right.$$
$$\left. - \sum_{a,b=\mathrm{f,g}} \sigma^{ab} e_{ii}^{a}(\mathbf{u})e_{jj}^{b}(\mathbf{u})\right]^{1/2} \left[\sigma_{\mathrm{M}}\left(t_{ij}^{\mathrm{s}}(\mathbf{u})t_{ij}^{\mathrm{s}}(\mathbf{u}) + t_{ij}^{\mathrm{f}}(\mathbf{u})t_{ij}^{\mathrm{f}}(\mathbf{u}) + t_{ij}^{\mathrm{g}}(\mathbf{u})t_{ij}^{\mathrm{g}}(\mathbf{u})\right)\right]^{1/2}$$
$$+ \left[\lambda_{v}\dot{e}_{ii}^{\mathrm{f}}(\mathbf{u})\dot{e}_{jj}^{\mathrm{f}}(\mathbf{u}) + 2\mu_{v}\dot{e}_{ij}^{\mathrm{f}}(\mathbf{u})\dot{e}_{ij}^{\mathrm{f}}(\mathbf{u})\right]^{1/2} \left[\mu_{\mathrm{M}}^{v} t_{ij}^{\mathrm{f}}(\mathbf{u})t_{ij}^{\mathrm{f}}(\mathbf{u})\right]^{1/2}, \tag{16}$$

so that, by (9), we get

$$\sum_{\alpha=\mathrm{s,f,g}} t_{ij}^{\alpha}(\mathbf{u})t_{ij}^{\alpha}(\mathbf{u}) \leqslant \left\{[2\sigma_{\mathrm{M}}\mathscr{E}(\mathbf{u})]^{1/2} + \left[\mu_{\mathrm{M}}^{v}\left(\lambda_{v}\dot{e}_{ii}^{\mathrm{f}}(\mathbf{u})\dot{e}_{jj}^{\mathrm{f}}(\mathbf{u}) + 2\mu_{v}\dot{e}_{ij}^{\mathrm{f}}(\mathbf{u})\dot{e}_{ij}^{\mathrm{f}}(\mathbf{u})\right)\right]^{1/2}\right\}^{2}. \tag{17}$$

Using the arithmetic–geometric mean inequality

$$a_1 a_2 \leqslant \frac{\epsilon_1}{2}a_1^2 + \frac{1}{2\epsilon_1}a_2^2, \tag{18}$$

which holds for every $\epsilon_1 > 0$ and every a_1 and a_2, we obtain the relation (14) and the proof is complete. □

By using the fact that the symmetric matrix $\mathbf{\Delta}$ is positive semi-definite, we obtain the following:

Corollary 1. *If* $\mathscr{P}$ *is an admissible process for the mixture satisfying the constitutive equation* (2), *then for every positive* ϵ_1, *we have*

$$\sum_{\alpha=\mathrm{s,f,g}} s_i^{\alpha}s_i^{\alpha} \leqslant 2(1+\epsilon_1)\sigma_{\mathrm{M}}\mathscr{E}(\mathbf{u}) + \left(1 + \frac{1}{\epsilon_1}\right)\mu_{\mathrm{M}}^{v}\mathbf{\Phi}(\mathbf{u}). \tag{19}$$

3. Spatial behavior for bounded and unbounded bodies

In this section we establish some estimates describing spatial decay and growth properties for dynamic processes associated with the mixture. We define and establish some properties of the time-weighted surface power function associated with the dynamic process $\mathscr{P}$, then we derive the mentioned results.

We consider a given time interval $[0,T]$, $T \in [0,\infty)$. Given the dynamic process $\mathscr{P}$, corresponding to the external force system $\mathscr{F}$, we introduce the set $\widehat{D}_T$ of all points in $\overline{B}$ so that:

(i) if $\mathbf{x} \in B$ then

$$u_i^{\alpha}(\mathbf{x},0) \neq 0 \quad \text{or} \quad \dot{u}_i^{\alpha}(\mathbf{x},0) \neq 0 \quad \text{for some } \alpha \in \{\mathrm{s,f,g}\}, \tag{20}$$

or

$$f_i^{\mathrm{s}}(\mathbf{x},\tau) \neq 0 \quad \text{or} \quad f_i^{\mathrm{f}}(\mathbf{x},\tau) \neq 0 \quad \text{or} \quad f_i^{\mathrm{g}}(\mathbf{x},\tau) \neq 0 \quad \text{for some } \tau \in [0,T], \tag{21}$$

(ii) if $\mathbf{x} \in \partial B$ then

$$\sum_{\alpha=\mathrm{s,f,g}} s_i^{\alpha}(\mathbf{x},\tau)\dot{u}_i^{\alpha}(\mathbf{x},\tau) \neq 0 \quad \text{for some } \tau \in [0,T]. \tag{22}$$

Roughly speaking, $\widehat{D}_T$ represents the support of the initial and boundary data and the body forces on the time interval $[0, T]$. If the region B is unbounded, then we will assume that $\widehat{D}_T$ is a bounded region.

We consider next a nonempty set $\widehat{D}_T^\star$ of $\overline{B}$ such that $\widehat{D}_T \subset \widehat{D}_T^\star \subset \overline{B}$ and

(1) If $\widehat{D}_T \cap B \neq \emptyset$, we choose $\widehat{D}_T^\star$ to be the smallest regular region in $\overline{B}$ that includes $\widehat{D}_T$; in particular, we set $\widehat{D}_T^\star = \widehat{D}_T$ if $\widehat{D}_T$ happens to be a regular region;
(2) If $\emptyset \neq \widehat{D}_T \subset \partial B$, we choose $\widehat{D}_T^\star$ to be the smallest regular subsurface of ∂B that includes $\widehat{D}_T$; in particular, we set $\widehat{D}_T^\star = \widehat{D}_T$ if $\widehat{D}_T$ is a regular subsurface of ∂B;
(3) If $\widehat{D}_T$ is empty, then we choose $\widehat{D}_T^\star$ to be an arbitrary regular subsurface of ∂B.

On this basis we introduce the set D_r, $r \geqslant 0$ by

$$D_r = \left\{ \mathbf{x} \in \overline{B};\ \widehat{D}_T^\star \cap \overline{\Sigma(\mathbf{x}, r)} \neq \emptyset \right\}, \tag{23}$$

where $\Sigma(\mathbf{x}, r)$ is the open ball with radius r and center $\mathbf{x}$. We shall use the notation B_r for the part of B contained in $B \setminus D_r$ and we set $B(r_1, r_2) = B_{r_2} \setminus B_{r_1}$, $r_1 > r_2$. Moreover, we shall denote by S_r the subsurface of ∂B_r contained into inside of B and whose outward unit normal vector is forwarded to the exterior of D_r.

We note that for a bounded body r ranges over $[0, L]$, where

$$L = \max\{\min\{[(x_i - y_i)(x_i - y_i)]^{1/2} : \mathbf{y} \in \widehat{D}_T^\star\} : \mathbf{x} \in \overline{B}\}. \tag{24}$$

We associate with the dynamic process $\mathscr{P}$ the following time-weighted surface power function $Q(r, t)$ defined by

$$Q(r, t) = -\int_0^t \int_{S_r} \mathrm{e}^{-\gamma z} \sum_{\alpha=\mathrm{s,f,g}} s_i^\alpha(z) \dot{u}_i^\alpha(z)\, \mathrm{d}a\, \mathrm{d}z, \quad r \geqslant 0,\ \ t \in [0, T], \tag{25}$$

where s_i^α, $\alpha = \mathrm{s, f, g}$ are defined by the relation (5) and γ is a prescribed positive parameter. Further, we introduce, for later convenience, the notation

$$\widetilde{Q}(r, t) = \int_0^t Q(r, z)\, \mathrm{d}z. \tag{26}$$

The next theorem shows a set of properties of the time-weighted surface power function that are useful in the study of the spatial behavior of the dynamic processes. The results give rise to counterparts of the thermoelastic versions established by Chiriţă and Ciarletta (1999).

Theorem 1 (Properties of the time-weighted surface power function Q). *Let* $\mathscr{P} = \{\mathbf{u}^\mathrm{s}, \mathbf{u}^\mathrm{f}, \mathbf{u}^\mathrm{g};\ \mathbf{e}^\mathrm{s}, \mathbf{e}^\mathrm{f}, \mathbf{e}^\mathrm{g};\ \mathbf{t}^\mathrm{s}, \mathbf{t}^\mathrm{f}, \mathbf{t}^\mathrm{g};\ \mathbf{p}^\mathrm{f}, \mathbf{p}^\mathrm{g}\}$ *be a dynamic process for the mixture on* $\overline{B}$ *corresponding to the external force system* $\mathscr{F}$ *and let* $\widehat{D}_T$ *be the bounded support of the corresponding data on the time interval* $[0, T]$. *Then the time-weighted surface power function* $Q(r, t)$ *has the following properties*:

(Q_1) *For each* $t \in [0, T]$ *and* $0 \leqslant r_2 \leqslant r_1$

$$\begin{aligned} Q(r_1, t) - Q(r_2, t) = &-\int_{B(r_1, r_2)} \mathrm{e}^{-\gamma t} \left[\frac{1}{2} \sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(t) \dot{u}_i^\alpha(t) + \mathscr{E}(\mathbf{u}(t)) \right] \mathrm{d}v \\ &- \int_0^t \int_{B(r_1, r_2)} \mathrm{e}^{-\gamma z} \left\{ \gamma \left[\frac{1}{2} \sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(z) \dot{u}_i^\alpha(z) + \mathscr{E}(\mathbf{u}(z)) \right] + \mathbf{\Phi}(\mathbf{u}(z)) \right\} \mathrm{d}v\, \mathrm{d}z; \end{aligned} \tag{27}$$

(Q_2) *$Q(r,t)$ is a continuous differentiable function on $r \geqslant 0$, $t \in [0,T]$, and*

$$\frac{\partial}{\partial r}Q(r,t) = -\int_{S_r} \mathrm{e}^{-\gamma t}\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(t)\dot{u}_i^\alpha(t) + \mathscr{E}(\mathbf{u}(t))\right]\mathrm{d}a - \int_0^t \int_{S_r} \mathrm{e}^{-\gamma z}\left\{\gamma\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z) + \mathscr{E}(\mathbf{u}(z))\right] + \mathbf{\Phi}(\mathbf{u}(z))\right\}\mathrm{d}a\,\mathrm{d}z; \tag{28}$$

(Q_3) *For each fixed $t \in [0,T]$, $Q(r,t)$ and $\widetilde{Q}(r,t)$ are non-increasing functions with respect to r;*
(Q_4) *$Q(r,t)$ satisfies the following first-order differential inequality:*

$$\frac{\gamma}{c}|Q(r,t)| + \frac{\partial}{\partial r}Q(r,t) \leqslant 0, \quad r \geqslant 0, \ t \in [0,T], \tag{29}$$

where

$$c = \sqrt{\frac{2\sigma_\mathrm{M} + \gamma\mu_\mathrm{M}^v}{2\rho_0}}, \qquad \rho_0 = \min\{\rho_0^\mathrm{s}, \rho_0^\mathrm{f}, \rho_0^\mathrm{g}\}; \tag{30}$$

(Q_5) *$\widetilde{Q}(r,t)$ satisfies the following first-order differential inequality:*

$$\sqrt{t}k(t)\frac{\partial}{\partial r}\widetilde{Q}(r,t) + |\widetilde{Q}(r,t)| \leqslant 0, \tag{31}$$

where

$$k(t) = \sqrt{\frac{2t\sigma_\mathrm{M} + \mu_\mathrm{M}^v}{2\rho_0}}; \tag{32}$$

(Q_6) *If B is a bounded body, then $Q(r,t)$ and $\widetilde{Q}(r,t)$ are positive functions.*

Proof. In view of the relations (2)–(5), (25), the definition of $\widehat{D}_T$ and the divergence theorem we get

$$Q(r_1,t) - Q(r_2,t) = -\int_0^t \int_{\partial B(r_1,r_2)} \mathrm{e}^{-\gamma z}\sum_{\alpha=\mathrm{s,f,g}} s_i^\alpha(z)\dot{u}_i^\alpha(z)\,\mathrm{d}a\,\mathrm{d}z = -\int_0^t \int_{B(r_1,r_2)} \mathrm{e}^{-\gamma z}\sum_{\alpha=\mathrm{s,f,g}}\left[t_{ji,j}^\alpha(z)\dot{u}_i^\alpha(z) + t_{ij}^\alpha(z)\dot{e}_{ij}^\alpha(z)\right]\mathrm{d}v\,\mathrm{d}z. \tag{33}$$

Further, we use the basic equations (1)–(3) and the relations (8) and (9) in order to obtain

$$Q(r_1,t) - Q(r_2,t) = -\int_0^t \int_{B(r_1,r_2)} \mathrm{e}^{-\gamma z}\left\{\frac{\partial}{\partial z}\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z) + \mathscr{E}(\mathbf{u}(z))\right] + \mathbf{\Phi}(\mathbf{u}(z))\right\}\mathrm{d}v\,\mathrm{d}z, \tag{34}$$

which by means of an integration by parts and the definition of $\widehat{D}_T$ leads to the identity (27), and so the part (Q_1) is established.

Part (Q_2) and (Q_3) follows from the definition of the dynamic process and the property (Q_1).

We now establish the property (Q_4). On the basis of the Schwarz's inequality, the arithmetic-geometric mean inequality (18) and Corollary 1, from (25), we obtain

$$
\begin{aligned}
|Q(r,t)| &\leqslant \int_0^t \int_{S_r} e^{-\gamma z}\left|\sum_{\alpha=s,f,g} s_i^\alpha(z)\dot{u}_i^\alpha(z)\right| \mathrm{d}a\,\mathrm{d}z \\
&\leqslant \int_0^t \int_{S_r} e^{-\gamma z}\left[\frac{\epsilon_2}{2}\sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z) + \frac{1}{2\epsilon_2\rho_0}\sum_{\alpha=s,f,g} s_i^\alpha(z)s_i^\alpha(z)\right]\mathrm{d}a\,\mathrm{d}z \\
&\leqslant \int_0^t \int_{Sr} e^{-\gamma z}\left\{\frac{\epsilon_2}{2\gamma}\sum_{\alpha=s,f,g}\gamma\rho_0^\alpha \dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z) + \frac{1}{2\epsilon_2\rho_0\gamma}\left[2\gamma(1+\epsilon_3)\sigma_M\mathscr{E}(\mathbf{u}(z))\right.\right. \\
&\qquad \left.\left. + \gamma\left(1+\frac{1}{\epsilon_3}\right)\mu_M^v\mathbf{\Phi}(\mathbf{u}(z))\right]\right\}\mathrm{d}a\,\mathrm{d}z, \quad \forall\epsilon_2>0,\ \epsilon_3>0,
\end{aligned}
\tag{35}
$$

where ρ_0 is defined by the second term of $(30)_2$. Now we equate the coefficients of the various energetic terms in the last integral in (35), that is, we set

$$
\frac{\epsilon_2}{\gamma} = \frac{(1+\epsilon_3)\sigma_M}{\epsilon_2\rho_0\gamma} = \frac{(1+1/\epsilon_3)\mu_M^v}{2\epsilon_2\rho_0}. \tag{36}
$$

Therefore, we choose for the arbitrary parameters ϵ_2 and ϵ_3 the following values:

$$
\epsilon_2 = c, \qquad \epsilon_3 = \frac{\gamma\mu_M^v}{2\sigma_M}, \tag{37}
$$

where c is given by the first term of $(30)_1$. With these choices substituted in (35) and by using the relation (28) we deduce the first-order differential inequality (29).

We consider now the part (Q_5). By means of the Schwarz's inequality and the relation

$$
\int_0^t \int_0^z g^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}z \leqslant t\int_0^t g^2(z)\,\mathrm{d}z, \tag{38}
$$

from (26), we get

$$
\begin{aligned}
|\widetilde{Q}(r,t)| &= \left|\int_0^t\int_0^z\int_{S_r} e^{-\gamma\tau}\sum_{\alpha=s,f,g} s_i^\alpha(\tau)\dot{u}_i^\alpha(\tau)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}z\right| \leqslant \left(\sqrt{t}\int_0^t\int_{S_r} e^{-\gamma z}\sum_{\alpha=s,f,g}\rho_0^\alpha\dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z)\,\mathrm{d}a\,\mathrm{d}z\right)^{1/2} \\
&\qquad \times\left(\sqrt{t}\int_0^t\int_0^z\int_{S_r} e^{-\gamma\tau}\frac{1}{\rho_0}\sum_{\alpha=s,f,g} s_i^\alpha(\tau)s_i^\alpha(\tau)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}z\right)^{1/2}.
\end{aligned}
\tag{39}
$$

Moreover, we use the arithmetic–geometric mean inequality and the Corollary 1 in (39) in order to obtain

$$
\begin{aligned}
|\widetilde{Q}(r,t)| &\leqslant \sqrt{t}\left\{\int_0^t\int_{Sr} e^{-\gamma z}\frac{\epsilon_4}{2}\sum_{\alpha=s,f,g}\rho_0^\alpha\dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z)\,\mathrm{d}a\,\mathrm{d}z\right. \\
&\qquad \left. + \int_0^t\int_0^z\int_{S_r} e^{-\gamma\tau}\frac{1}{2\epsilon_4\rho_0}\left[2(1+\epsilon_5)\sigma_M\mathscr{E}(\mathbf{u}(\tau)) + \left(1+\frac{1}{\epsilon_5}\right)\mu_M^v\mathbf{\Phi}(\mathbf{u}(\tau))\right]\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}z\right\} \\
&\leqslant \sqrt{t}\left\{\int_0^t\int_{S_r} e^{-\gamma z}\left[\frac{\epsilon_4}{2}\sum_{\alpha=s,f,g}\rho_0^\alpha\dot{u}_i^\alpha(z)\dot{u}_i^\alpha(z) + \frac{t(1+\epsilon_5)\sigma_M}{\epsilon_4\rho_0}\mathscr{E}(\mathbf{u}(z))\right]\mathrm{d}a\,\mathrm{d}z\right. \\
&\qquad \left. + \frac{(1+1/\epsilon_5)\mu_M^v}{2\epsilon_4\rho_0}\int_0^t\int_0^z\int_{S_r} e^{-\gamma\tau}[\mathbf{\Phi}(\mathbf{u}(\tau))]\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}z\right\}, \quad \forall\epsilon_4>0,\ \epsilon_5>0.
\end{aligned}
\tag{40}
$$

Then, we equate the coefficients of the various energetic terms in the last inequality of the relation (40), that is we set

$$\epsilon_4 = k(t), \qquad \epsilon_5 = \frac{\mu_{\mathrm{M}}^{v}}{2t\sigma_{\mathrm{M}}}, \tag{41}$$

where $k(t)$ is given by the relation (32). If we combine the inequality (40) with the choices (41) and the property (Q_2) we obtain the estimate (31). Thus, the property (Q_5) is established.

We proceed now to establish (Q_6). The definition of $\widehat{D}_T$ together with the relations (24) and (25) then show that

$$Q(L,t) = 0, \quad t \in [0,T]. \tag{42}$$

The part (Q_6) follows now from the relations (26) and (27) by means of the use of the relation (42). Thus, the proof is complete. □

Theorem 2 (Spatial behavior for bounded bodies). *Let* $\mathscr{P} = \{\mathbf{u}^{\mathrm{s}}, \mathbf{u}^{\mathrm{f}}, \mathbf{u}^{\mathrm{g}};\ \mathbf{e}^{\mathrm{s}}, \mathbf{e}^{\mathrm{f}}, \mathbf{e}^{\mathrm{g}};\ \mathbf{t}^{\mathrm{s}}, \mathbf{t}^{\mathrm{f}}, \mathbf{t}^{\mathrm{g}};\ \mathbf{p}^{\mathrm{f}}, \mathbf{p}^{\mathrm{g}}\}$ *be a dynamic process for the mixture on the bounded regular region* $\overline{B}$, *corresponding to the external force system* $\mathscr{F} = \{\mathbf{f}^{\mathrm{s}}, \mathbf{f}^{\mathrm{f}}, \mathbf{f}^{\mathrm{g}};\ \mathbf{s}^{\mathrm{s}}, \mathbf{s}^{\mathrm{f}}, \mathbf{s}^{\mathrm{g}}\}$. *Suppose that the external given data have the bounded support* $\widehat{D}_T$ *on the time interval* $[0,T]$ *and let* $Q(r,t)$ *be the time-weighted surface power function associated with* $\mathscr{P}$. *Then, for each fixed* $t \in [0,T]$, *we have*

$$Q(r,t) \leqslant Q(0,t)\exp\left(-\frac{\gamma}{c}r\right), \quad r \in [0,L] \tag{43}$$

and

$$\widetilde{Q}(r,t) \leqslant \widetilde{Q}(0,t)\exp\left(-\frac{1}{\sqrt{t}k(t)}r\right), \quad r \in [0,L], \tag{44}$$

where c *and* $k(t)$ *are given by* (30) *and* (32), *respectively*.

Proof. By means of the property (Q_6) the relations (29) and (31) can be written in the following form:

$$\frac{\partial}{\partial r}\left[\exp\left(\frac{\gamma}{c}r\right)Q(r,t)\right] \leqslant 0, \quad r \in [0,L], \tag{45}$$

$$\frac{\partial}{\partial r}\left[\exp\left(\frac{1}{\sqrt{t}k(t)}r\right)\widetilde{Q}(r,t)\right] \leqslant 0, \quad r \in [0,L]. \tag{46}$$

By an integration with respect to r, we obtain the estimates (43) and (44). Thus, the proof is complete. □

We now consider an unbounded body so that we assume that B is an unbounded regular region. We derive some results of Phragmèn–Lindelöf type as it is described in the next theorem.

Theorem 3 (Spatial behavior for unbounded bodies). *Let* $\mathscr{P} = \{\mathbf{u}^{\mathrm{s}}, \mathbf{u}^{\mathrm{f}}, \mathbf{u}^{\mathrm{g}};\ \mathbf{e}^{\mathrm{s}}, \mathbf{e}^{\mathrm{f}}, \mathbf{e}^{\mathrm{g}};\ \mathbf{t}^{\mathrm{s}}, \mathbf{t}^{\mathrm{f}}, \mathbf{t}^{\mathrm{g}};\ \mathbf{p}^{\mathrm{f}}, \mathbf{p}^{\mathrm{g}}\}$ *be a dynamic process for the mixture on the unbounded region* $\overline{B}$, *corresponding to the external force system* $\mathscr{F} = \{\mathbf{f}^{\mathrm{s}}, \mathbf{f}^{\mathrm{f}}, \mathbf{f}^{\mathrm{g}};\ \mathbf{s}^{\mathrm{s}}, \mathbf{s}^{\mathrm{f}}, \mathbf{s}^{\mathrm{g}}\}$. *Suppose that the external given data have the bounded support* $\widehat{D}_T$ *on the time interval* $[0,T]$ *and let* $Q(r,t)$ *be the time-weighted surface power function associated with* $\mathscr{P}$. *Then, for each fixed* $t \in [0,T]$, *the following alternative holds*:

(i) *either* $Q(r,t) \geqslant 0$ *for all* $r \geqslant 0$ *and then*

$$Q(r,t) \leqslant Q(0,t)\exp\left(-\frac{\gamma}{c}r\right), \quad r \geqslant 0, \tag{47}$$

or

(ii) *there exists a value* $r_t \geqslant 0$ *so that* $Q(r_t, t) < 0$ *and then* $Q(r, t) < 0$ *for all* $r \geqslant r_t$ *and*

$$-Q(r,t) \geqslant -Q(r_t,t)\exp\left(\frac{\gamma}{c}(r-r_t)\right), \quad r \geqslant r_t. \tag{48}$$

Proof. Let t be fixed in $[0, T]$. Then it results, from the property (Q_3), that we have only the following two possibilities:

(a) $Q(r,t) \geqslant 0$ for all $r \in [0,\infty)$;
(b) there exists $r_t \in [0,\infty)$ such that $Q(r_t,t) < 0$.

Let us consider the first possibility; that is, we assume $Q(r,t) \geqslant 0$ for all $r \in [0,\infty)$. Then the differential inequality (29) can be written in the form (45) and so we get the estimate (47).

Let us now consider the case (b). Then, we have $Q(r,t) \leqslant Q(r_t,t) < 0$ for all $r \geqslant r_t$ so that the differential inequality (29) implies that

$$-\frac{\gamma}{c}Q(r,t) + \frac{\partial}{\partial r}Q(r,t) \leqslant 0, \quad r \geqslant r_t. \tag{49}$$

Thus, by an integration, from (49), we obtain the estimate (48) and the proof is complete. □

A similar argument with that in the above proves the following theorem:

Theorem 4. *Suppose the hypotheses of Theorem* 3 *hold true. Then, for each* $t \in [0, T]$, *either*

(i) $\widetilde{Q}(r,t) \geqslant 0$ *for all* $r \geqslant 0$ *and then*

$$\widetilde{Q}(r,t) \leqslant \widetilde{Q}(0,t)\exp\left(-\frac{1}{\sqrt{tk(t)}}r\right), \quad r \geqslant 0, \tag{50}$$

or

(ii) *there exists a value* $r_t^\star \geqslant 0$ *so that* $\widetilde{Q}(r_t^\star, t) < 0$ *and then* $\widetilde{Q}(r,t) < 0$ *for all* $r \geqslant r_t^\star$ *and*

$$-\widetilde{Q}(r,t) \geqslant -\widetilde{Q}(r_t^\star,t)\exp\left(\frac{1}{\sqrt{tk(t)}}(r-r_t^\star)\right), \quad r \geqslant r_t^\star. \tag{51}$$

Remark 1. It is worth to mention that one can obtain a good description for the spatial behavior of dynamic processes by combining the results described by the estimates (43) and (44) or, alternatively, the estimates (47) and (50). The decay estimates (44) and (50) are useful for short values of time, while the decay estimates (43) and (47) give a good description for large values of time.

4. Further asymptotic spatial behavior

In this section we outline a class of mixtures for which we can complete the study of spatial behavior with some results describing the asymptotic spatial behavior of the processes. In this aim we adopt a volumetric measure for the processes derived from (25) and then we establish some spatial decay estimates by using qualitative methods involving second-order partial differential inequalities. The results give rise to counterparts of thermoelastic versions of those established by Horgan et al. (1984) for the transient heat conduction and Scalia (2002) for materials with voids.

We assume that B is a semi-infinite cylinder and we choose the rectangular Cartesian system so that the generators of the cylinder are parallel with the x_3-axis and the end of cylinder is contained in the plane $x_3 = -l$, $l > 0$. Further, we suppose that the support of the external given data $\widehat{D}_T$ is enclosed in the half–space $x_3 < 0$. Throughout in what follows we assume that

$$\dot{u}_i^{\mathrm{f}} = 0 \quad \text{on} \quad (\partial B_0 - S_0) \times [0, T]. \tag{52}$$

We note that the constitutive equations (2) allow us to write

$$t_{ij}^{\mathrm{f}} = \tilde{t}_{ij}^{\mathrm{f}} + \bar{t}_{ij}^{\mathrm{f}}, \tag{53}$$

where

$$\tilde{t}_{ij}^{\mathrm{f}} = -\left(\sigma^{\mathrm{f}} e_{rr}^{\mathrm{s}} + \sum_{a=\mathrm{f,g}} \sigma^{\mathrm{f}a} e_{rr}^{a}\right)\delta_{ij},$$

$$\bar{t}_{ij}^{\mathrm{f}} = \lambda_v \dot{e}_{rr}^{\mathrm{f}} \delta_{ij} + 2\mu_v \dot{e}_{ij}^{\mathrm{f}}. \tag{54}$$

It is easy to see that the procedure used to obtain the estimate (14) can be applied to obtain the estimate

$$\sum_{\alpha=\mathrm{s,g}} t_{ij}^{\alpha}(u) t_{ij}^{\alpha}(u) + \tilde{t}_{ij}^{\mathrm{f}}(u)\tilde{t}_{ij}^{\mathrm{f}}(u) \leqslant 2\sigma_{\mathrm{M}} \mathscr{E}(u). \tag{55}$$

Within the above context the time-weighted surface power function $Q(r, t)$ defined by the relation (25) becomes

$$\begin{aligned} Q(r,t) = &-\int_0^t \int_{S_r} \mathrm{e}^{-\gamma z} \left[\sum_{\alpha=\mathrm{s,g}} t_{3i}^{\alpha}(z)\dot{u}_i^{\alpha}(z) + \tilde{t}_{3i}^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z)\right] \mathrm{d}a\,\mathrm{d}z \\ &- \int_0^t \int_{S_r} \mathrm{e}^{-\gamma z} \bar{t}_{3i}^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z)\,\mathrm{d}a\,\mathrm{d}z, \quad r \in [0,\infty), \;\; t \in [0,T]. \end{aligned} \tag{56}$$

By using the relations (52), $(54)_2$ and the divergence theorem, we obtain

$$\int_{S_r} \bar{t}_{3i}^{\mathrm{f}} \dot{u}_i^{\mathrm{f}}\,\mathrm{d}a = \int_{S_r} (\lambda_v - \mu_v)\dot{u}_3 \dot{u}_{\rho,\rho}^{\mathrm{f}}\,\mathrm{d}a + \frac{1}{2}\left\{\int_{S_r}\left[\mu_v \dot{u}_i^{\mathrm{f}}\dot{u}_i^{\mathrm{f}} + (\lambda_v + \mu_v)\dot{u}_3^{\mathrm{f}}\dot{u}_3^{\mathrm{f}}\right]\mathrm{d}a\right\}_{,3}. \tag{57}$$

It follows from the relation (57) that in the class of fluids for which

$$\lambda_v = \mu_v, \tag{58}$$

the function $Q(r, t)$ can be written as

$$\begin{aligned} Q(r,t) = &-\int_0^t \int_{S_r} \mathrm{e}^{-\gamma z} \left[\sum_{\alpha=\mathrm{s,g}} t_{3i}^{\alpha}(z)\dot{u}_i^{\alpha}(z) + \tilde{t}_{3i}^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z)\right] \mathrm{d}a\,\mathrm{d}z \\ &- \frac{\mu_v}{2}\left\{\int_0^t \int_{S_r} \mathrm{e}^{-\gamma z}\left[\dot{u}_i^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z) + 2\dot{u}_3^{\mathrm{f}}(z)\dot{u}_3^{\mathrm{f}}(z)\right]\mathrm{d}a\,\mathrm{d}z\right\}_{,3}. \end{aligned} \tag{59}$$

We denote by $\mathscr{M}$ the class of processes $\mathscr{P}$ for which $Q(r, t) \geqslant 0$ for $r \in [0, \infty)$, $t \in [0, T]$ and (52) is satisfied. For the processes $\mathscr{P}$ residing in the set $\mathscr{M}$ it follows by Theorem 3 that the estimate (47) holds true. Thus, we can introduce the following measure:

$$I(r,t) = \int_r^{\infty} Q(\xi, t)\,\mathrm{d}\xi, \quad r \in [0,\infty), \;\; t \in [0,T], \tag{60}$$

that is

$$I(r,t) = -\int_0^t \int_{B_r} \mathrm{e}^{-\gamma z}\left[\sum_{\alpha=\mathrm{s,g}} t_{3i}^{\alpha}(z)\dot{u}_i^{\alpha}(z) + \tilde{t}_{3i}^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z)\right]\mathrm{d}v\,\mathrm{d}z + \frac{\mu_v}{2}\int_0^t \int_{S_r} \mathrm{e}^{-\gamma z}\left[\dot{u}_i^{\mathrm{f}}(z)\dot{u}_i^{\mathrm{f}}(z) + 2\dot{u}_3^{\mathrm{f}}(z)\dot{u}_3^{\mathrm{f}}(z)\right]\mathrm{d}a\,\mathrm{d}z. \tag{61}$$

Lemma 2. *Let $\mathscr{P} \in \mathscr{M}$ be a dynamic process, associated with a mixture for which $\lambda_v = \mu_v$, on the semi–infinite cylinder B, corresponding to external given data $\mathscr{F}$. Then, the volumetric measure $I(r,t)$ satisfies the following second-order partial differential inequality*

$$\frac{\partial I}{\partial t}(r,t) \leqslant -c_1\frac{\partial I}{\partial r}(r,t) + a\frac{\partial^2 I}{\partial r^2}(r,t), \quad r \in [0,\infty),\ \ t \in [0,T], \tag{62}$$

where

$$c_1 = \sqrt{\frac{\sigma_{\mathrm{M}}}{\rho_0}}, \qquad a = \frac{3\mu_v}{\rho_0^{\mathrm{f}}}. \tag{63}$$

Proof. By a direct differentiation in (61), we deduce that

$$\frac{\partial I}{\partial t}(r,t) = -\int_{B_r} \mathrm{e}^{-\gamma t}\left[\sum_{\alpha=\mathrm{s,g}} t_{3i}^{\alpha}(t)\dot{u}_i^{\alpha}(t) + \tilde{t}_{3i}^{\mathrm{f}}(t)\dot{u}_i^{\mathrm{f}}(t)\right]\mathrm{d}v\,\mathrm{d}z + \frac{\mu_v}{2}\int_{S_r} \mathrm{e}^{-\gamma t}\left[\dot{u}_i^{\mathrm{f}}(t)\dot{u}_i^{\mathrm{f}}(t) + 2\dot{u}_3^{\mathrm{f}}(t)\dot{u}_3^{\mathrm{f}}(t)\right]\mathrm{d}a\,\mathrm{d}z. \tag{64}$$

Further, the relation (47) gives

$$\lim_{r\to\infty} Q(r,t) = 0, \tag{65}$$

so that, by means of the relations (27), (28), and (60), we obtain

$$\frac{\partial I}{\partial r}(r,t) = -Q(r,t) = -\int_{B_r} \mathrm{e}^{-\gamma t}\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^{\alpha}\dot{u}_i^{\alpha}(t)\dot{u}_i^{\alpha}(t) + \mathscr{E}(\mathbf{u}(t))\right]\mathrm{d}v - \int_0^t \int_{B_r} \mathrm{e}^{-\gamma z}\left\{\gamma\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^{\alpha}\dot{u}_i^{\alpha}(z)\dot{u}_i^{\alpha}(z) + \mathscr{E}(\mathbf{u}(z))\right] + \mathbf{\Phi}(\mathbf{u}(z))\right\}\mathrm{d}v\,\mathrm{d}z, \tag{66}$$

and

$$\frac{\partial^2 I}{\partial r^2}(r,t) = -\frac{\partial Q}{\partial r}(r,t) = \int_{S_r} \mathrm{e}^{-\gamma t}\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^{\alpha}\dot{u}_i^{\alpha}(t)\dot{u}_i^{\alpha}(t) + \mathscr{E}(\mathbf{u}(t))\right]\mathrm{d}a + \int_0^t \int_{S_r} \mathrm{e}^{-\gamma z}\left\{\gamma\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^{\alpha}\dot{u}_i^{\alpha}(z)\dot{u}_i^{\alpha}(z) + \mathscr{E}(\mathbf{u}(z))\right] + \mathbf{\Phi}(\mathbf{u}(z))\right\}\mathrm{d}a\,\mathrm{d}z. \tag{67}$$

At this time we use Schwarz's inequality and the arithmetic–geometric mean inequality in order to get

$$\frac{\partial I}{\partial t}(r,t) \leqslant \int_{B_r} \mathrm{e}^{-\gamma t}\left[\frac{\epsilon_6}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(t)\dot{u}_i^\alpha(t) + \frac{1}{2\rho_0\epsilon_6}\left(\sum_{\alpha=\mathrm{s,g}} t_{3i}^\alpha(t)t_{3i}^\alpha(t) + \tilde{t}_{3i}^{\mathrm{f}}(t)\tilde{t}_{3i}^{\mathrm{f}}(t)\right)\right]\mathrm{d}v + \frac{3\mu_v}{2}\int_{S_r} \mathrm{e}^{-\gamma t}\dot{u}_i^{\mathrm{f}}(t)\dot{u}_i^{\mathrm{f}}(t)\,\mathrm{d}a, \quad \epsilon_6 > 0. \tag{68}$$

If we set

$$\epsilon_6 = c_1, \tag{69}$$

from the relation (55), we deduce

$$\frac{\partial I}{\partial t}(r,t) \leqslant c_1\int_{B_r} \mathrm{e}^{-\gamma t}\left[\frac{1}{2}\sum_{\alpha=\mathrm{s,f,g}} \rho_0^\alpha \dot{u}_i^\alpha(t)\dot{u}_i^\alpha(t) + \mathscr{E}(\mathbf{u}(t))\right]\mathrm{d}v + \frac{a}{2}\int_{S_r} \mathrm{e}^{-\gamma t}\rho_0^{\mathrm{f}}\dot{u}_i^{\mathrm{f}}(t)\dot{u}_i^{\mathrm{f}}(t)\,\mathrm{d}a. \tag{70}$$

By taking into account the relations (66) and (67) in (70), we obtain the inequality (62) and the proof is complete. □

Theorem 5 (Asymptotic spatial behavior). *Let $\mathscr{P} \in \mathscr{M}$ be a dynamic process, associated with a mixture for which $\lambda_v = \mu_v$, on the semi–infinite cylinder B, corresponding to external given data $\mathscr{F}$. Then, for each fixed $t \in [0,T]$, we have*

$$I(r,t) \leqslant \left(\max_{z\in[0,t]} I(0,z)\right)\exp\left\{\frac{c_1}{2a}r\right\}G(r,t), \tag{71}$$

where

$$G(r,t) = \frac{1}{2\sqrt{a\pi}}\int_0^t rz^{-3/2}\exp\left\{-\left(\frac{r^2}{4az} + \frac{c_1^2}{4a}z\right)\right\}\mathrm{d}z. \tag{72}$$

Proof. If we make the following change of function:

$$I(r,t) = \exp\left\{-\frac{c_1^2}{4a}t\right\}\exp\left\{\frac{c_1}{2a}r\right\}J(r,t), \tag{73}$$

then, we can write the relation (62) in the form

$$\frac{\partial J}{\partial t}(r,t) \leqslant a\frac{\partial^2 J}{\partial r^2}(r,t), \quad r \in [0,\infty), \ t \in [0,T]. \tag{74}$$

It follows from the relations (60), (61), (73) and (74) that $J(r,t)$ satisfies

$$\begin{aligned} & a\frac{\partial^2 J}{\partial r^2}(r,t) - \frac{\partial J}{\partial t}(r,t) \geqslant 0, \quad r \in [0,\infty), \ t \in [0,T], \\ & J(r,0) = 0, \quad r \in [0,\infty) \\ & J(0,t) = \exp\left\{\frac{c_1^2}{4a}t\right\}I(0,t) \geqslant 0, \quad t \in [0,T], \\ & J(r,t) \to 0 \quad (\text{uniformly in } t) \text{ as } r \to \infty. \end{aligned} \tag{75}$$

By using the maximum principle for parabolic differential equations (see Protter and Weinberger, 1967), we get

$$J(r,t) \leqslant w(r,t), \quad r \in [0,\infty), \ t \in [0,T], \tag{76}$$

where $w(r,t)$ is the solution for the following one-dimensional heat equation

$$\begin{aligned}
&a\frac{\partial^2 w}{\partial r^2}(r,t) - \frac{\partial w}{\partial t}(r,t) = 0, \quad r \in [0,\infty), \ t \in [0,T],\\
&w(r,0) = 0, \quad r \in [0,\infty)\\
&w(0,t) = \exp\left\{\frac{c_1^2}{4a}t\right\} I(0,t) \geqslant 0, \quad t \in [0,T],\\
&w(r,t) \to 0 \quad (\text{uniformly in } t) \text{ as } r \to \infty.
\end{aligned} \tag{77}$$

The solution of such a problem is given by Tikhonov and Samarskii (1964, p. 208)

$$w(r,t) = \frac{a}{2\sqrt{\pi}} \int_0^t \frac{r}{[a(t-z)]^{3/2}} \exp\left\{-\frac{r^2}{4a(t-z)}\right\} \exp\left\{\frac{c_1^2}{4a}z\right\} I(0,z)\,\mathrm{d}z. \tag{78}$$

From the relations (73), (76) and (78), we obtain the estimate (71) and the proof is complete. □

From the relation (71) we can obtain various estimates for $I(r,t)$ by using estimates for the function $G(r,t)$. Some estimates for a function like $G(r,t)$ are established by Horgan et al. (1984). Using the estimate deduced by Scalia (2002)

$$G(r,t) \leqslant \frac{2r(at/\pi)^{1/2} \exp\{-(c_1^2/4a)t\}}{r^2 - c_1^2 t^2} \exp\left\{-\frac{r^2}{4at}\right\}, \quad \text{for } r > c_1 t, \tag{79}$$

we obtain the following.

Theorem 6. *Assume the hypotheses of Theorem 5 hold true. Then, for each $r \in (0,\infty)$, $r > c_1 t$, we have*

$$I(r,t) \leqslant \left(\max_{z\in[0,t]} I(0,z)\right) \frac{1}{r^2 - c_1^2 t^2} 2r\left(\frac{at}{\pi}\right)^{1/2} \exp\left\{-\frac{c_1^2}{4a}t\right\} \exp\left(\frac{c_1 r}{2a} - \frac{r^2}{4at}\right). \tag{80}$$

For $r > c_1 t$, the spatial decay estimate (80) proves explicitly that, for all fixed $t \in [0,T]$, the decay rate is controlled by the factor $\exp(-r^2/4at)$ for large distances to the support of the external given data. Such a decay rate is similar to that found by Horgan et al. (1984) for heat conduction equation and Scalia (2002) for thermoelastic materials with voids.

At large distances of the support of the external given data, the spatial decay of processes is influenced only by the coefficients of the fluid.

In the theory of mixtures of elastic solids asymptotic behavior of solutions were studied by Dafermos (1976), Martinez and Quintanilla (1995) and spatial behavior by Pompei and Scalia (1999).

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